

## Modular Arithmetic Intro I

Note 6

**Modular arithmetic:** working number “mod  $m$ ”: restrict to only  $\{0, 1, \dots, m - 1\}$ ; other numbers are *equivalent* to some number in this set.

Think of a clock; “13-o’clock” is the same as “1-o’clock”;  $2m$  is the same as  $m$ , which is the same as 0. We can go the other way too; 0 is the same as  $-m$ , etc.

Saying  $a \equiv b \pmod{m}$  means:

- $a, b$  have same remainder when divided by  $m$
- $a = b + km$  for some integer  $k$
- $m \mid (a - b)$

**Operations:** Suppose  $a \equiv b \pmod{m}$  and  $c \equiv d \pmod{m}$ .

Addition/subtraction:  $a \pm c \equiv b \pm d \pmod{m}$

Multiplication:  $ac \equiv bd \pmod{m}$

Exponentiation:  $a^k \equiv b^k \pmod{m}$ . You *cannot* apply the mod to the exponent.

There is no division; there are multiplicative inverses though:  $x^{-1} \equiv a \pmod{m}$  means  $ax \equiv 1 \pmod{m}$ .

## 1 Party Tricks

Note 6

You are at a party celebrating your completion of the CS 70 midterm. Show off your modular arithmetic skills and impress your friends by quickly figuring out the last digit(s) of each of the following numbers:

(a) Find the last digit of  $11^{3142}$ .

(b) Find the last digit of  $9^{9999}$ .

(c) Find the last digit of  $3^{641}$ .

## 2 Modular Potpourri

Note 6

Prove or disprove the following statements:

(a) There exists some  $x \in \mathbb{Z}$  such that  $x \equiv 3 \pmod{16}$  and  $x \equiv 4 \pmod{6}$ .

(b)  $2x \equiv 4 \pmod{12} \iff x \equiv 2 \pmod{12}$ .

(c)  $2x \equiv 4 \pmod{12} \iff x \equiv 2 \pmod{6}$ .

## 3 Modular Inverses

Note 6

Recall the definition of inverses from lecture: let  $a, m \in \mathbb{Z}$  and  $m > 0$ ; if  $x \in \mathbb{Z}$  satisfies  $ax \equiv 1 \pmod{m}$ , then we say  $x$  is an **inverse of  $a$  modulo  $m$** .

Now, we will investigate the existence and uniqueness of inverses.

(a) Is 3 an inverse of 5 modulo 14?

(b) Is 3 an inverse of 5 modulo 10?

(c) For all  $n \in \mathbb{N}$ , is  $3 + 14n$  an inverse of 5 modulo 14?

(d) Does 4 have an inverse modulo 8?

(e) Suppose  $x, x' \in \mathbb{Z}$  are both inverses of  $a$  modulo  $m$ . Is it possible that  $x \not\equiv x' \pmod{m}$ ?

## 4 Fibonacci GCD

### Note 6

(a) Prove that, for positive integers  $x$  and  $y$ , if  $x \geq y$ , then  $\gcd(x, y) = \gcd(y, x - y)$ .

(b) The Fibonacci sequence is given by  $F_n = F_{n-1} + F_{n-2}$ , where  $F_0 = 0$  and  $F_1 = 1$ . Prove that, for all  $n \geq 1$ ,  $\gcd(F_n, F_{n-1}) = 1$ .