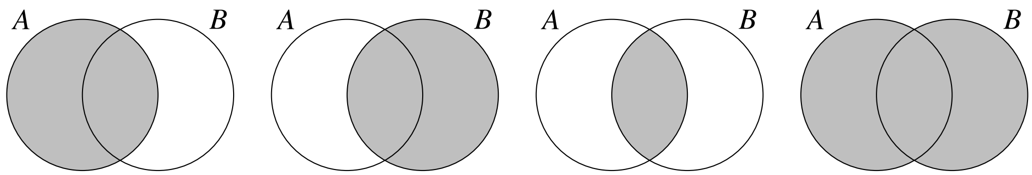
CS 70 Discrete Mathematics and Probability Theory
Spring 2026 Sinclair, Song

DIS 7B

Counting Intro II

Note 12

Inclusion-exclusion: With two sets,



$$|A| \quad + \quad |B| \quad - \quad |A \cap B| \quad = \quad |A \cup B|$$

With more sets,

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| = &|A_1| + |A_2| + \cdots + |A_n| \\ &- |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_i \cap A_j| - \cdots - |A_{n-1} \cap A_n| \\ &+ |A_1 \cap A_2 \cap A_3| + \cdots + |A_i \cap A_j \cap A_k| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| \\ &\cdots\end{aligned}$$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=k}} \left|\bigcap_{i \in S} A_i\right|$$

That is, for each size k, iterate through all ways of picking k sets from $\{A_1, \ldots, A_n\}$, and alternate between adding and subtracting the sizes of their intersection.

Combinatorial proofs: A technique for proving combinatorial identities. There should be very little math involved (usually none): use two different ways of counting the same scenario. One way should correspond to the left-hand side of the equality, and the other way should correspond to the right-hand side of the equality. The fact that we're counting the same scenario means that the two sides are equal.

1 Inclusion and Exclusion

Note 12

What is the total number of natural numbers strictly less than 100 that are also coprime to 100?

2 CS70: The Musical

Note 12

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(b) Edward would now like to select a crew out of n people. Use this to provide a combinatorial argument that proves the following identity: (this is called Pascal's Identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:

$$\sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} = 2^{n-j} \binom{n}{j}.$$