CS 70 Spring 2026	Discrete Mathematics and Probability Theory Sinclair, Song	HW 01

Due: Saturday, 1/31, 4:00 PM
Grace period until Saturday, 1/31, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Logical Equivalence?

Note 1

Decide whether each of the following logical equivalences is correct and justify your answer.

(a) $\forall x\, (P(x) \land Q(x)) \stackrel{?}{\equiv} \forall x\, P(x) \land \forall x\, Q(x)$

(b) $\forall x\, (P(x) \lor Q(x)) \stackrel{?}{\equiv} \forall x\, P(x) \lor \forall x\, Q(x)$

(c) $\exists x\, (P(x) \lor Q(x)) \stackrel{?}{\equiv} \exists x\, P(x) \lor \exists x\, Q(x)$

(d) $\exists x\, (P(x) \land Q(x)) \stackrel{?}{\equiv} \exists x\, P(x) \land \exists x\, Q(x)$

2 Prove

Note 2

Prove each of the following statements

(a) $\forall a,b,c \in \mathbb{Z}$, if $a|b$ and $b|c$, then $a|c$.

(b) $\forall n \in \mathbb{N}$, n is odd if and only if $3n+5$ is even

(c) $\forall n \in \mathbb{Z}$, n^2+n+6 is even.

3 Prove or Disprove

Note 2

For each of the following, either prove the statement, or disprove by finding a counterexample.

(a) $(\forall n \in \mathbb{N})$ if n is odd then n^2+4n is odd.

(b) $(\forall a,b \in \mathbb{R})$ if $a+b \leq 15$ then $a \leq 11$ or $b \leq 4$.

(c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.

(d) $(\forall n \in \mathbb{Z}^+)$ $5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers)

(e) The product of a non-zero rational number and an irrational number is irrational.

4 Induction Starter

Note 3

Consider the inequality $2^n < n!$ (the right hand side is a factorial, not an exclamation mark).

(a) Make a conjecture as to which $n \in \mathbb{N}$ will have the inequality hold.

We will now prove your conjecture using induction.

(b) What is your base case?

(c) What is the inductive hypothesis for this proof?

(d) What do we want to show in the inductive step?

(e) Conclude the proof with the inductive step.

5 A Coin Game

Note 3

Your "friend" Stanley Ford suggests you play the following game with him. You each start with a single stack of n coins. On each of your turns, you select one of your stacks of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be $3 \cdot 2 = 6$). You continue taking turns until all your stacks have only one coin in them. Stan then plays the same game with his stack of n coins, and whoever ends up with the largest total score over all their turns wins.

Prove that no matter how you choose to split the stacks, your total score will always be $\frac{n(n-1)}{2}$. (This means that you and Stan will end up with the same score no matter what happens, so the game is rather pointless.)

6 Proving Inequality

Note 3

For all positive integers $n \geq 1$, prove with induction that

$$\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} < \frac{1}{2}.$$

(Note: while you can use formula for an infinite geometric series to prove this, we require you to use induction. If direct induction seems difficult, consider strengthening the inductive hypothesis. Can you prove an equality statement instead of an inequality?)

7 Self-Grades

Make sure to review the self grades post on Edstem and submit your selfgrades for the previous homework assignment on Gradescope! This is just a reminder to do so, no need to submit anything for this question.