CS 70 Discrete Mathematics and Probability Theory
Spring 2026 Sinclair, Song

HW 06

Due: Saturday, 3/7, 4:00 PM
Grace period until Saturday, 3/7, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Unions and Intersections

Note 10

Given:

- X is a countable, non-empty set. For all $i \in X$, A_i is an uncountable set.
- Y is an uncountable set. For all $i \in Y$, B_i is a countable set.

For each of the following, decide if the expression is "Always countable", "Always uncountable", "Sometimes countable, Sometimes uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a) $X \cap Y$

(b) $X \cup Y$

(c) $\bigcup_{i \in X} A_i$

(d) $\bigcap_{i \in X} A_i$

(e) $\bigcup_{i \in Y} B_i$

(f) $\bigcap_{i \in Y} B_i$

2 Count It!

Note 10

For each of the following sets, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

(a) The integers which divide 8.

(b) The integers which 8 divides.

(c) The functions from $\mathbb{N}$ to $\mathbb{N}$.

(d) The set of strings over the English alphabet. (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.)

(e) The set of finite-length strings drawn from a countably infinite alphabet, $\mathscr{A}$.

(f) The set of infinite-length strings over the English alphabet.

3 Unprogrammable Programs

Note 11

Prove whether the programs described below can exist or not.

(a) A program $P(F,x,y)$ that returns true if the program F outputs y when given x as input (i.e. $F(x) = y$) and false otherwise.

(b) A program P that takes two programs F and G as arguments, and returns true if for all inputs x, F halts on x iff G halts on x (and returns false if this equivalence is not always true).

Hint: Use P to solve the halting problem, and consider defining two subroutines to pass in to P, where one of the subroutines always loops.

4 Computations on Programs

Note 11

(a) Is it possible to write a program that takes a natural number n as input, and finds the shortest arithmetic formula which computes n? For the purpose of this question, a formula is a sequence consisting of some valid combination of (decimal) digits, standard binary operators ($+$, $\times$, the "^" operator that raises to a power), and parentheses. We define the length of a formula as the number of characters in the formula. Specifically, each operator, decimal digit, or parentheses counts as one character.

(*Hint:* Think about whether it's possible to enumerate the set of possible arithmetic formulas. How would you know when to stop?)

(b) Now say you wish to write a program that, given a natural number input n, finds another program (e.g. in Java or C) which prints out n. The discovered program should have the minimum execution-time-plus-length of all the programs that print n. Execution time is measured by the number of CPU instructions executed, while "length" is the number of characters in the source code. Can this be done?

(*Hint:* Is it possible to tell whether a program halts on a given input within t steps? What can you say about the execution-time-plus-length of the program if you know that it does not halt within t steps?)