

Due: Saturday, 4/18, 4:00 PM
Grace period until Saturday, 4/18, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Coupon Collector Variance

Note 15
Note 17

It's that time of the year again (again)—Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of n different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

Let X be the number of visits you have to make before you can redeem the grand prize. Show that

$$\text{Var}(X) = n^2 \left(\sum_{i=1}^n \frac{1}{i^2} \right) - \mathbb{E}[X]$$

.

2 Double-Check Your Intuition Again

Note 17

- (a) You roll a fair six-sided die and record the result X . You roll the die again and record the result Y .
- What is $\text{cov}(X + Y, X - Y)$?
 - Prove that $X + Y$ and $X - Y$ are not independent.

The problems below are not related to the scenario above. For each of them, if you think the answer is "yes" then provide a proof. If you think the answer is "no", then provide a counterexample.

- If X is a random variable and $\text{Var}(X) = 0$, then must X be a constant?
- If X is a random variable and c is a constant, then is $\text{Var}(cX) = c \text{Var}(X)$?
- If A and B are random variables with nonzero standard deviations and $\text{Corr}(A, B) = 0$, then are A and B independent?

- (e) If X and Y are not necessarily independent random variables, but $\text{Corr}(X, Y) = 0$, and X and Y have nonzero standard deviations, then is $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$?

The two subparts below are **optional** and will not be graded but are recommended for practice.

- (f) If X and Y are random variables then is $\mathbb{E}[\max(X, Y) \min(X, Y)] = \mathbb{E}[XY]$?
- (g) If X and Y are independent random variables with nonzero standard deviations, then is

$$\text{Corr}(\max(X, Y), \min(X, Y)) = \text{Corr}(X, Y)?$$

3 Dice Games

Note 17

- (a) Alice rolls a fair six-sided die until she gets a 1. Let X be the number of total rolls she makes (including the last one), and let Y be the number of rolls on which she gets an even number. Compute $\mathbb{E}[Y | X = x]$, and use it to calculate $\mathbb{E}[Y]$.
- (b) Bob plays a game in which he starts off with one fair six-sided die. At each time step, he rolls all the dice he has. Then, for each die, if it comes up as an odd number, he puts that die back, and adds a number of fair six-sided dice equal to the number displayed to his collection. (For example, if he rolls a one on the first time step, he puts that die back along with an extra die.) However, if it comes up as an even number, he removes that die from his collection.

Compute the expected number of dice Bob will have after n time steps. (Hint: compute the value of $\mathbb{E}[X_k | X_{k-1} = m]$ to derive a recursive expression for X_k , where X_i is the random variable representing the number of dice after i time steps. As your base case, $E[X_0] = 1$.)

4 Fishy Computations

Note 18

Assume for each part that the random variable can be modelled by a Poisson distribution.

- (a) Suppose that on average, a fisherman catches 20 salmon per week. What is the probability that he will catch exactly 7 salmon this week?
- (b) Suppose that on average, you go to Fisherman's Wharf twice a year. What is the probability that you will go at most once in 2024?
- (c) Suppose on average, there are 5.7 boats that sail in Laguna Beach per day. What is the probability there will be *at least* 3 boats sailing throughout the *next two days* in Laguna?

5 Geometric and Poisson

Note 18

Let $X \sim \text{Geometric}(p)$ and $Y \sim \text{Poisson}(\lambda)$ be independent random variables. Compute $\mathbb{P}[X > Y]$. Your final answer should not have summations.

Hint: Use the total probability rule.

6 Balls and Bins

Note 17

Throw n balls into m bins, where m and n are positive integers. Let X be the number of bins with exactly one ball. Compute $\text{Var}(X)$. Your final answer should not contain any summations.